Int. J. Solids Structures, 1975, Vol. 11, pp. 21–31. Pergamon Press. Printed in Gt. Britain.

APPLICATION OF HILL FUNCTIONS TO TWO-DIMENSIONAL PLATE PROBLEMS†

ROBERT KAO‡
Department of Civil and Mechanical Engineering,
The Catholic University of America, Washington, D.C. 20017, U.S.A.

(*Received* 9 *January* 1974; *revised* 10 *April* 1974)

Abstract—Hill functions which are constructed on the basis of Legendre polynomials are used as coordinate functions to solve two-dimensional plate problems. In order to effectively handle various boundary conditions and without introducing additional equations to the original system of equations, the method of artificial parameters is employed for this purpose. Illustrative examples are performed and numerical results obtained are compared very nicely with theoretical solutions in the literature.

INTRODUCTION

In application of the Rayleigh–Ritz method to obtain numerical solutions for given problems, important points are on the selection of coordinate functions and on the treatment of boundary conditions. The selection of coordinate functions should be such that they must be able to provide good accuracy of solutions and also to make the evaluation of matrix elements of system equations relatively easier; various boundaries should be handled with ease regardless of which types of coordinate functions are employed.

In this paper, a particular type of function—hill functions[1–4]—is used as coordinate function and a powerful method, namely, the method of artificial parameters, is utilized to deal with various boundary conditions. The functions in their given domains are first divided into a number of portions and the functions in each portion are then contsructed on the local coordinate system on the basis of Legendre polynomials. The advantages of using these functions are that evaluation of stiffness matrix elements of system equations will simply turn out to be some kind of multiplication of coefficients of hill functions and their corresponding derivatives, and that good accuracy of numerical results is generally obtained.

Applications of hill functions to one-dimensional problems have been reported in Refs. [1, 5]. In this paper, these functions are, to the knowledge of the author, for the first time applied to solve two-dimensional problems.

RAYLEIGH–RITZ METHOD

For the sake of completeness and the use in the following sections, key equations involved in the Rayleigh–Ritz method are presented in this section.

Suppose there exists a sequence $y_1, y_2, \ldots$ of admissible functions in the variational problem such that

$$\lim_{n\to\infty} F(y_n) = d \tag{1}$$

† The research reported in here was supported by the Office of Naval Research (Contract NR 064-452).
‡ Research Assistant Professor.

where d is the lower bound of the functional $F(y)$. The Rayleigh–Ritz method is a recipe for the construction of such a sequence by choosing an arbitrary system of coordinate functions, $\omega_1, \omega_2, \ldots,$ with the property that any linear combination

$$y_n = c_1\omega_1 + c_2\omega_2 + \cdots + c_n\omega_n \tag{2}$$

is admissible in the variational problem, and that the solution function y and its relevant derivatives may be approximated with any degree of accuracy by equation (2) and its corresponding derivatives, respectively.

If the problem under consideration, $F(y)$, is a quadratic functional, then the values c_i can be determined by n linear simultaneous equations

$$\frac{\partial F(y_n)}{\partial c_i} = 0, \qquad i = 1, 2, \ldots, n. \tag{3}$$

COORDINATE FUNCTIONS—HILL FUNCTIONS

Some finite element models called "hill functions" have recently been developed in the applied mathematics community[1–4]. These models, which can hardly satisfy given boundary conditions without special considerations but are quite economic when used in numerical computation, are utilized as coordinate functions in this paper.

The construction of hill functions has been described in fair detail in Refs. [1, 2], here only an outline of the equations is given:

$$^n\phi(x) = \sum_{j=1}^{n} {}^n\phi_j(\xi), \qquad -n/2 \le x \le n/2 \tag{4a}$$

$$^n\phi_j(\xi) = \sum_{i=1}^{n} {}^n\alpha_{i,j} P_i(\xi), \qquad -1/2 \le \xi \le 1/2 \tag{4b}$$

where $^n\phi(x)$ denotes the hill function of order n; x and ξ represent the global and local coordinate systems, respectively.

The graphical representation of equations (4) are sketched for $n = 4$ and 5 in Fig. 1. From this figure, it is noted that the entire domain of $^n\phi(x)$, $-n/2 \le x \le n/2$, is divided into n even intervals, and in each interval a local coordinate system $(-1/2 \le \xi \le 1/2)$ is set up having the origin at the interval center. Thus, $^n\phi(x)$ can be taken as a sum of n portions, equation (4a), and each portion of this function is represented in the local coordinate system by a Fourier series expansion in terms of Legendre polynomials $P_i(\xi)$ (with $P_1(\xi) = 1$), equation (4b) ($^n\alpha_{i,j}$ in this equation are coefficient constants).

Derivative expressions of hill functions may also be given as follows:

$$^n\phi_1^{(k)}(\xi) = {}^{n-1}\phi_1^{(k-1)}(\xi), \qquad {}^n\phi_n^{(k)}(\xi) = -{}^{n-1}\phi_{n-1}^{(k-1)}(\xi),$$

$$^n\phi_j^{(k)}(\xi) = {}^{n-1}\phi_j^{(k-1)}(\xi) - {}^{n-1}\phi_{j-1}^{(k-1)}(\xi) \quad \text{for} \quad j = 2, \ldots, n-1,$$

or

$$^n\phi_j^{(k)}(\xi) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} {}^{n-k}\phi_{j-i}(\xi) \tag{5a}$$

$$(k = 1, \ldots, n-2; j = 1, \ldots, n; 1 \le j - i \le n - k).$$

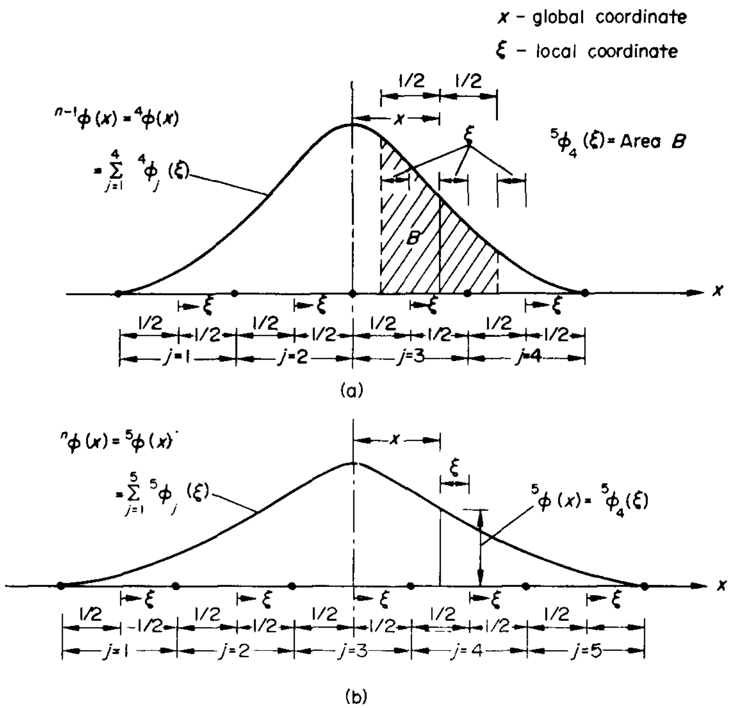

Fig. 1

and

$$^n\phi_j^{(k)}(\xi) = \sum_{i=1}^{n-k} {}^n b_{i,j}^{(k)} P_i(\xi), \qquad j = 1, 2, \ldots, n. \tag{5b}$$

where the superscript (k) denotes the order of derivatives. The coefficients $^n b_{i,j}^{(k)}$ can be obtained in terms of $^{n-k}a_{i,j}$ through equation (5a).

The numerical values of $^n a_{i,j}$ with n ranging from 1 to 4 is given in Table 1; the hill functions corresponding to these values are plotted in Fig. 2. Incidentally, a computer program has been developed computing $^n a_{i,j}$ and $^n b_{i,j}^{(k)}$ up to any order as desired.

Table 1.† $^n a_{i,j}$ in equations (4) ($n = 1, \ldots, 4$)

n	j	$i=1$	$i=2$	$i=3$	$i=4$
1	1	1			
2	1	0·5	0·5		
3	1	0·1667	0·25	0·08333	
	2	0·6667	0	−0·1666	
4	1	0·04167	0·075	0·04167	0·008333
	2	0·4583	0·275	−0·04167	−0·025

† Because of symmetry of hill functions about $x = 0$, only symmetric parts of coefficients are given; for more accurate values, a double precision version may be used in the computer code.

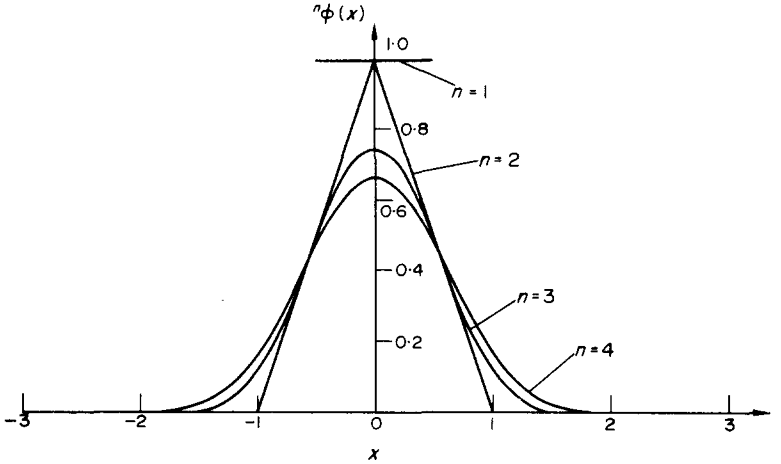

Fig. 2

PROBLEM FORMULATION

The governing differential equation for rectangular plate problems (Fig. 3) has been derived in great detail in Ref.[6] and may be given as follows:

$$\frac{\partial^4 w}{\partial x^4} + 2\,\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D} \tag{6}$$

in which w is the normal displacement function, q is the distributed loading, and

$$D = \frac{Et^3}{12(1 - \nu^2)} \tag{7}$$

where E is the modulus of elasticity of the material, ν is Poisson's ratio and t is the thickness of the plate.

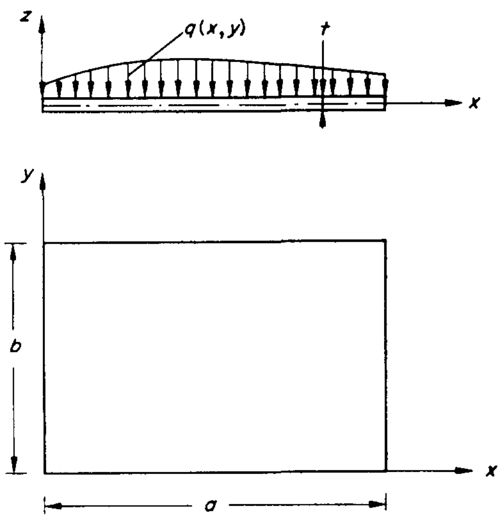

Fig. 3

The entire problem description may be completed by further providing with boundary conditions. In general, boundary conditions associated with each edge may be expressed in the form of

$$\begin{aligned} \Phi_1(w, w_x, w_y, \ldots) = 0 \\ \Phi_2(w, w_x, w_y, \ldots) = 0 \end{aligned} \tag{8}$$

where $(\ )_x = \partial(\ )/\partial x$. For a rectangular plate, there are four edges and hence a total of eight boundary conditions in the form of equations (8) need be considered.

In accordance with the Rayleigh–Ritz method, the entire problem must be cast into energy form; the energy expression associated with equation (6) is given in Ref. [6] by

$$F_1 = \iint \left(\frac{D}{2} \left\{ \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1 - \nu) \left[\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \, \partial y} \right)^2 \right] \right\} - wq \right) \mathrm{d}x \, \mathrm{d}y \tag{9}$$

If coordinate functions are chosen such that boundary conditions on four edges have been automatically satisfied, then an application of the Rayleigh–Ritz method to equation (9) will yield a desired result. However, for arbitrary chosen coordinate functions, such as the hill functions as described in the previous section, these boundary conditions will generally not be accommodated, and a special consideration to comply with these conditions becomes necessary.

The particular method elected here to deal with boundary conditions is the method of artificial parameters[1, 7]. According to this method, two types of artificial (parameter) springs, β and γ, are intentionally introduced at each edge of the plate. The energy contributions due to these parameters are added to equation (9) to form the total energy for the entire system:

$$\begin{aligned} F = \iint \left(\frac{D}{2} \left\{ \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1 - \nu) \left[\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \, \partial y} \right)^2 \right] \right\} - wq \right) \mathrm{d}x \, \mathrm{d}y \\ + \tfrac{1}{2}(\beta w^2 + \gamma w_x^2)|_{x=0,a} + \tfrac{1}{2}(\beta w^2 + \gamma w_y^2)|_{y=0,b}. \end{aligned} \tag{10}$$

By applying variational principle to equation (10), it can be shown for each edge of the plate: (a) if $\beta, \gamma = 0$ are assigned for an edge, this will yield a free boundary condition; (b) if $\beta = \infty$ and $\gamma = 0$, this will give a simply supported condition; (c) if $\beta, \gamma = \infty$, this will produce a clamped edge condition. For a detail description of how the method of artificial parameters relates to different types of boundary conditions, readers should consult Refs. [1, 7].

In numerical analysis, β and γ cannot be increased to infinitive and, in fact, need be increased only to be large enough to approximate the fixity. In this connection, an observation based on results obtained by using varying values of β and γ tends to indicate that results are not so sensitive to the magnitude of these values. This observation is also evidenced from results obtained for one-dimensional problems[1].

The proper values for β and γ can easily be determined from experimenting on some simple cases and then using these values whenever they are required for all other calculations. As to the merits of using this particular method to handle various boundary conditions, particularly when compared with using other popular methods, an assessment will be given in the final section.

SYSTEM EQUATIONS

In this paper, hill functions as presented in the previous section are used as coordinate functions for all calculations. In order to fit dimension requirements as specified in hill functions, following nondimensional quantities are introduced:

$$\begin{aligned} &w = h\bar{w}, \qquad &x = h_1\bar{x}, \qquad &y = h_2\bar{y} \\ &h_1 = a/m_1, \qquad &h_2 = b/m_2, \qquad &h = (h_1 + h_2)/2 \end{aligned} \tag{11}$$

in which m_1 and m_2 are the total numbers of intervals, and h_1 and h_2 are their corresponding sizes of the intervals in x and y directions, respectively (Fig. 4). For simplicity in presentation, we discuss only the case of

$$h = h_1 = h_2 \tag{12}$$

while it is recognized that there should be no more difficulty for the case of $h_1 \neq h_2$ although some minor complexity is expected.

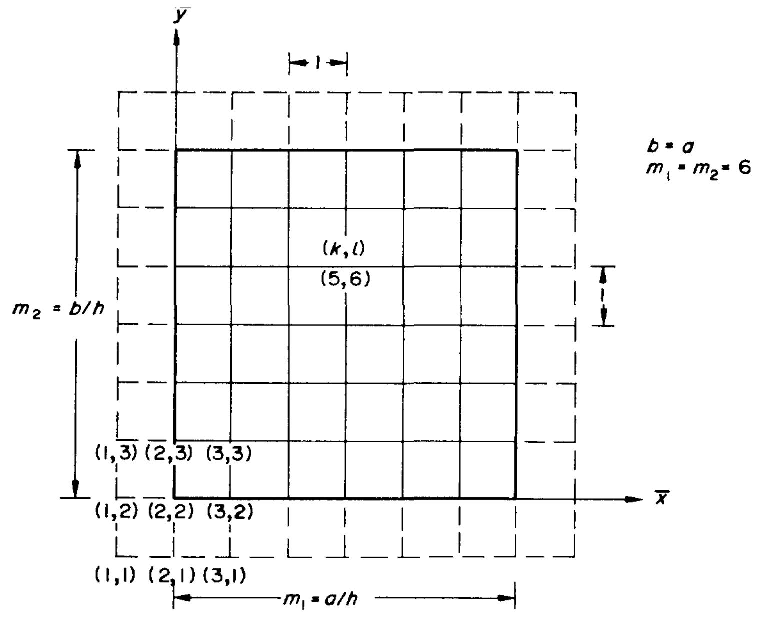

Fig. 4

Now, in the light of the Rayleigh–Ritz method, the nondimensional displacement function $\bar{w}$ can be expressed in terms of hill function of order n as

$$\bar{w}(\bar{x}, \bar{y}) = \sum c_{kl}\,{}^n\phi(\bar{x} - k)\,{}^n\phi(\bar{y} - l) \tag{13}$$

where c_{kl} is undetermined constant and (k, l) denotes a two-dimensional point as shown in Fig. 4. For easier graphical representation, a one-dimensional example,

$$\bar{w}(\bar{x}) = \sum c_k\,{}^n\phi(\bar{x} - k) \tag{14}$$

is shown in Fig. 5 for hill function of order $n = 4$ and a total mesh number of $m = 5$.

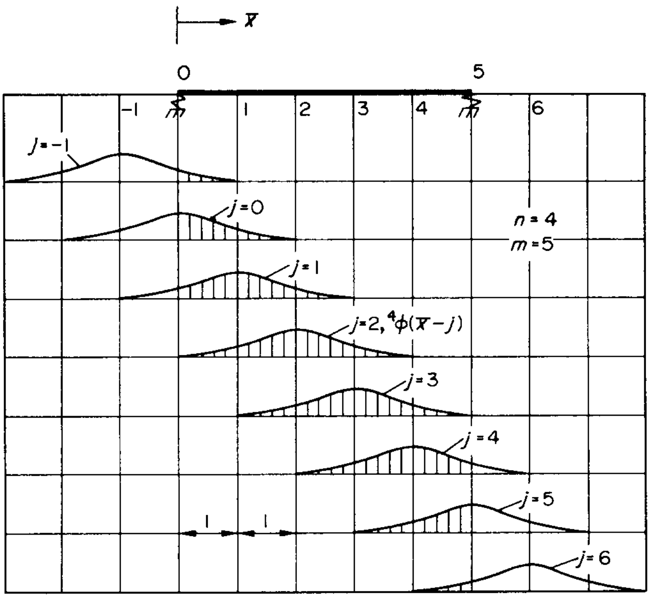

Fig. 5

Substitution of equation (13) into (10) yields a quadratic functional in c_{kl} and may be put into matrix form of

$$F = \tfrac{1}{2}\lfloor c\rfloor([K] + [K'])\{c\} - \lfloor c\rfloor\{q\} \tag{15}$$

where $[K]$ and $[K']$ are the stiffness matrix in connection with the system strain energy (equation (9) except the last term) and the matrix related to energy contributions from artificial springs, respectively; $\{q\}$ is a column matrix of loading terms. It is noticed that both $[K]$ and $[K']$ are symmetric matrices.

An application of equation (3) to equation (15) with variation of $\{c\}$ yields a system of equations:

$$([K] + [K'])\{c\} = \{q\} \tag{16}$$

For the purpose of references, a generic element of $[K]$ and $\{q\}$ may be given here:

$$\begin{aligned} K_{klst} = {} & [{}^n\phi''(\bar{x}-k)\,{}^n\phi(\bar{y}-l),\ {}^n\phi''(\bar{x}-s)\,{}^n\phi(\bar{y}-t)] \\ & + [{}^n\phi(\bar{x}-k)\,{}^n\phi''(\bar{y}-l),\ {}^n\phi(\bar{x}-s)\,{}^n\phi''(\bar{y}-t)] \\ & + \nu[{}^n\phi''(\bar{x}-k)\,{}^n\phi(\bar{y}-l),\ {}^n\phi(\bar{x}-s)\,{}^n\phi''(\bar{y}-t)] \\ & + \nu[{}^n\phi(\bar{x}-k)\,{}^n\phi''(\bar{y}-l),\ {}^n\phi''(\bar{x}-s)\,{}^n\phi(\bar{y}-t)] \\ & + 2(1-\nu)[{}^n\phi'(\bar{x}-k)\,{}^n\phi'(\bar{y}-l),\ {}^n\phi'(\bar{x}-s)\,{}^n\phi'(\bar{y}-t)] \end{aligned} \tag{17a}$$

$$q_{kl} = \frac{h^3}{D}\,[q(\bar{x},\bar{y}),\ {}^n\phi(\bar{x}-k)\,{}^n\phi(\bar{y}-l)] \tag{17b}$$

where

$$[A, B] = \int_{\bar{y}=0}^{m_2}\int_{\bar{x}=0}^{m_1} AB\,\mathrm{d}\bar{x}\,\mathrm{d}\bar{y}. \tag{18}$$

For example (see Fig. 4),

$$
\begin{aligned}
&[{}^n\phi''(\bar{x}-k){}^n\phi(\bar{y}-l), {}^n\phi''(\bar{x}-s){}^n\phi(\bar{y}-t)]_{k=2,\, l=2;\, s=3;\, t=4;\, n=4} \\
&= \int_0^{m_2} {}^n\phi(\bar{y}-l){}^n\phi(\bar{y}-t)\,\mathrm{d}\bar{y} \int_0^{m_1} {}^n\phi''(\bar{x}-k){}^n\phi''(\bar{x}-s)\,\mathrm{d}\bar{x} \\
&= \sum_{i=1}^{n=4} \frac{1}{2i-1}({}^4a_{i,3}{}^4a_{i,1} + {}^4a_{i,4}{}^4a_{i,2}) \\
&\times \sum_{i=1}^{n-2=2} \frac{1}{2i-1}({}^4b_{i,3}^{(2)}\,{}^4b_{i,2}^{(2)} + {}^4b_{i,4}^{(2)}\,{}^4b_{i,3}^{(2)}).
\end{aligned}
\tag{19}
$$

It should be mentioned here that, in arriving at the final expression for this integral, the evaluation is performed interval by interval, utilizing the orthogonality of the Legendre polynomials in $(-1/2, 1/2)$, i.e.

$$
\int_{-1/2}^{1/2} P_i(\xi)P_j(\xi)\,\mathrm{d}\xi = \begin{cases} 1/(2i-1), & i=j \\ 0 & , i \neq j. \end{cases}
\tag{20}
$$

It may also be noted that evaluation of equation (17b) does not enjoy such kind of simplicity and its value must be obtained numerically as follows: equation (17b) is first taken as a sum of integrals in each interval, and integrals in each interval are calculated by further breaking each interval into a number of subintervals and then employing usual numerical integration schemes.

NUMERICAL RESULTS

Three square plate problems with different combinations of boundary conditions are solved using equations (16) as system equations. In all calculations, a total number of meshes $m_1 = m_2 = 6$ and the hill function of order $n = 4$ are adopted (Fig. 4). Results are presented for the normal displacement w, and the moments along x and y directions, M_x and M_y. These moments are related to displacement function by

$$
M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right), \qquad M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right).
\tag{21}
$$

In solving equations (16), the Gauss–Jordan method [8] is employed using the largest element of the submatirx under consideration at each stage of operation to preserve the better numerical accuracy.

Case 1. *Simply supported square plate under uniform pressure*

In this case (Fig. 6), as discussed previously in Problem Formulation section, γ is set to be zero and β is made to be sufficiently large for all edges. Two different values of β are used and the results along the line $y = b/2$ are presented in Table 2. It is seen from this Table that the difference between the results corresponding to $\beta h^2/D = 10^2$ and 10^4 is about 1 per cent. Therefore, it may be appropriate to say that numerical results are not so sensitive to the magnitude of artificial parameters adopted. Presented also in this Table for the purpose of comparison are the Navier solutions as reported in Ref. [6].

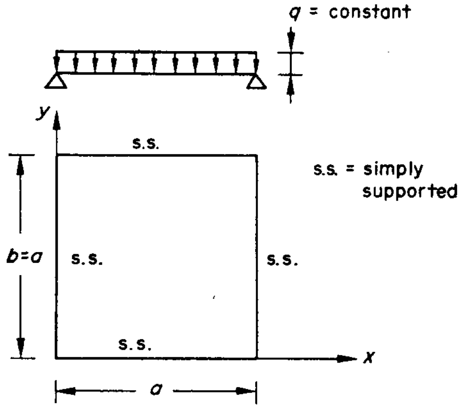

Fig. 6

Table 2. Comparison of present results and Navier solutions[6] (shown in parentheses) for a simply supported square plate under uniform pressure q (Fig. 6), hill function of order $n = 4$ and total number of meshes $m_1 = m_2 = 6$ are used, $y = b/2$, $\nu = 0{\cdot}3$

$\frac{\beta h^2}{D}$	$\frac{x}{a} = \frac{1}{6}$	$\frac{x}{a} = \frac{1}{3}$	$\frac{x}{a} = \frac{1}{2}$
	(a) Deflection wD/qa^4		
	(0·002107	0·003556	0·004062)
10^2	0·002132	0·003584	0·004092
10^4	0·002107	0·003555	0·004062
	(b) Moment along x – direction M_x/qa^2		
	(0·030547	0·044137	0·047886)
10^2	0·031978	0·044875	0·048422
10^4	0·031874	0·044757	0·048310
	(c) Moment along y – direction M_y/qa^2		
	(0·026200	0·042425	0·047886)
10^2	0·026853	0·042972	0·048422
10^4	0·026781	0·042870	0·048310

Case 2. *Uniformly loaded square plate with one edge built in and three others simply supported*

This case is shown in Fig. 7. In computation, $\gamma = 0$ and β is adjusted to be large enough for all three simply supported edges; both γ and β are assigned sufficiently large for the clamped edge. The present results using two different sets of values for β and γ along with series solutions as given in Ref.[6] are tabulated in Table 3.

Table 3. Comparison of present results and series solutions[6] (shown in parentheses) for a uniformly loaded square plate with on edge built in and three others simply supported (Fig. 7), $n = 4$, $m_1 = m_2 = 6$, $\nu = 0{\cdot}3$

$\frac{\beta h^2}{D}$	$\frac{\gamma}{D}$	w $x = y = a/2$	M_x $x = y = a/2$	M_y $x = y = a/2$	M_y $x = a/2, y = a$
		$(0{\cdot}0028qa^4/D$	$0{\cdot}034qa^4$	$0{\cdot}039qa^2$	$-0{\cdot}084qa^2)$
10^2	10^2	$0{\cdot}0028qa^4/D$	$0{\cdot}035qa^2$	$0{\cdot}040qa^2$	$-0{\cdot}079qa^2$
10^2	10^4	$0{\cdot}0028qa^4/D$	$0{\cdot}034qa^4$	$0{\cdot}040qa^2$	$-0{\cdot}079qa^2$

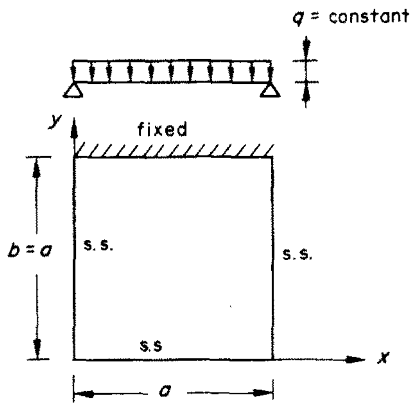

Fig. 7

Case 3. *Uniformly loaded square plate with one edge free and three others simply supported*

This situation is sketched in Fig. 8. The artificial spring parameters are arranged as

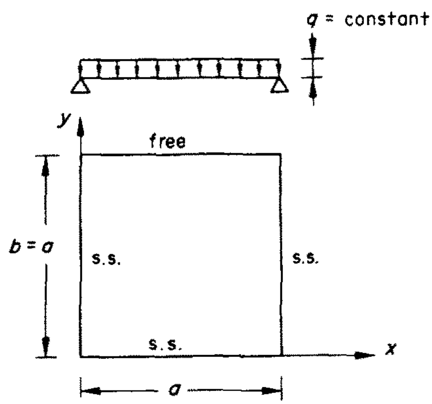

Fig. 8

follows: $\gamma = 0$ and β is set to be sufficiently large for all simply supported edges while $\beta = \gamma = 0$ for the free edge. The present results together with series solutions[6] are displayed in Table 4.

The numerical results in each case involving solving 81×81 matrix take about 80 seconds of PDP-10 CPU time.

Table 4. Comparison of present results and series solutions[6] (shown in parentheses) for a uniformly loaded square plate with one edge free and three others simply supported (Fig. 8), $n = 4$, $m_1 = m_2 = 6$, $\nu = 0{\cdot}3$

$\frac{\beta h^2}{D}$	w_{max} $x = a/2, y = a$	$(M_x)_{max}$ $x = a/2, y = a$	M_x $x = y = a/2$	M_y $x = y = a/2$
	$(0{\cdot}01286qa^4/D$	$0{\cdot}112qa^2$	$0{\cdot}080qa^2$	$0{\cdot}039qa^2)$
10^2	$0{\cdot}01289qa^4/D$	$0{\cdot}114qa^2$	$0{\cdot}081qa^2$	$0{\cdot}040qa^2$
10^4	$0{\cdot}01285qa^4/D$	$0{\cdot}114qa^2$	$0{\cdot}081qa^2$	$0{\cdot}040qa^2$

DISCUSSION AND CONCLUSIONS

Hill functions which are constructed on the basis of Legendre polynomials have been used as coordinate functions. Excellent results are obtained for deflection function and quite acceptable solutions are also produced for bending moments.

The reasoning of being able to yield good accuracy of numerical results by utilizing hill functions is probably from the fact that the unusual manner in which these functions are constructed. Each portion of hill functions is formed independently on the local coordinate system and, by so doing, this enables them to be constructed very accurately, or say, coefficients of hill functions as well as their corresponding derivatives can be calculated with very good precision.

Other factors which may also affect the accuracy of results are the number of meshes and the order of hill functions selected. In view of the nature of hill functions that they have continuous derivatives only up to the order of $n-2$, better results for bending moments in these case problems may also be obtained if higher order hill functions are employed; for plate problems considered herein, hill function of order $n = 6$ may very well serve this purpose.

It may be noted that since hill functions are extended over only a portion of the entire domain of the given problem, and the calculations are carried out interval by interval, the entire solution procedure described herein is sometimes referred to as finite element method [3, 4].

On the treatment of boundary conditions, one of the most popular techniques is the method of Lagrange multipliers which has been used in Ref. [5] to treat axisymmetric circular plate problems. In application of this method to the problems discussed in this paper, extra equations for boundary conditions will inevitably be introduced, the number of which will depend upon the total number of boundary points encountered. When the number of boundary points become large enough, this may cause a computer storage problem. On the other hand, the method of artificial parameters does not create any extra equations. Once the magnitudes of these parameters, which are not so sensitive to the numerical results obtained, have been determined, various boundary conditions can be quite well contained without giving any additional burden on the computer storage. Considering this important advantage, the method of artificial parameters should deserve more attention in its application to a variety of practical problems.

Finally, it may be concluded here that the hill function approach does indeed provide very good numerical results and also is quite economical in its application. Hence, it is recommended here that more research on its improvement as well as on its application to a wide range of problems should be pursued.

REFERENCES

1. R. Kao, Approximate Solutions by Utilizing Hill Functions, *Comp. Struc.* 3, 397–412 (1973).
2. J. Segethova, Numerical Construction of Hill Functions. *SIAM J. Numerical Analysis* 9, No. 2 199–204 (June 1972).
3. I. Babuska, The Finite Element Method for Elliptic Differential Equations, Lecture at Symposium on the numerical solution of partial differential equations, May 1970, University of Maryland.
4. M. H. Schultz, *Finite Element Analysis*, Department of Computer Science, Yale University (1971).
5. R. Kao, *Application of Hill Functions to Circular Plate Problems*, Technical Report, Department of Civil and Mechanical Engineering, The Catholic University of America, Washington, D.C. (1973).
6. S. Timoshenko and S. Woinowsky-Krieger, *Theory of Plates and Shells*, McGraw-Hill, New York (1959).
7. R. Courant, Variational Methods for the Solution of Problems of Equilibrium and Vibrations, *Bull. Am. Math. Soc.* **49**, 1–23 (1943).
8. J. M. McCormick and M. G. Salvadori, *Numerical Methods in Fortran*, Prentice Hall (1965).